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ON A GENERALIZATION OF A THEOREM OF B. TYLER

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ABSTRACT

Let B be an arbitrary normal matrix, satisfying some conditions. Absolute B-summability factors in a sequence for Cesàro method C^α if $\alpha \geqq 1$ or $\alpha = 0$ and absolute convergence factors in a sequence for C^α if $0 < \alpha < 1$ are obtained.

Let X and Y be two Banach spaces and ε_n be continuous linear operators from X to Y. Let $A = (a_{nk})$ be a triangular infinite matrix of complex numbers. For a sequence (U_n) where[†] $U_n \in X$ we denote

$$U'_n = \sum_{k=0}^{n} a_{nk} U_k \tag{1}$$

and

$$u'_n = \sum_{k=0}^{n} \bar{a}_{nk} U_k, \tag{2}$$

where $\bar{a}_{nk} = \bar{\Delta} a_{nk} = a_{nk} - a_{n-1,k}$ and $a_{-1,k} = 0$. The sequence (U_n) is called A-summable, if the limit $\lim U'_n$ exists in X. The sequence (U_n) is called absolutely A-summable or $|A|$-summable if the series $\Sigma \| u'_n \|$ is convergent. If $A = C^\alpha$ is the matrix of Cesàro method C^α of the order $\alpha \geqq 0$, then

$$a_{nk} = A^{\alpha-1}_{n-k} / A^\alpha_n, \quad A^\alpha_n = (n+\alpha)\cdots(1+\alpha)/n!, \quad A^\alpha_0 = 1.$$

Let also $B = (b_{nk})$ be a triangular infinite matrix of complex numbers. The operators $\varepsilon_n : X \to Y$ are called summability factors in a sequence of the type (A, B) if for any A-summable sequence (U_n) the sequence $(\varepsilon_n U_n)$ is B-summable. Summability factors in a sequence of the type $(|A|, |B|)$ are similarly defined. If $B = E$ is the unit matrix $E = (\delta_{nk})$ then summability factors

[†] Everywhere the free indices take on all values $0, 1, 2, \cdots$.

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are called convergence factors. To characterize the summability factors of a given type is to find effective (in practice, easily verifiable) necessary and sufficient conditions in order that ε_n should be summability factors of this type. In what follows, instead of saying that ε_n are summability factors in a sequence of the types (A, B) or $(|A|,|B|)$, we often write $\varepsilon_n \in (\mathfrak{A}, \mathfrak{B})$ or $\varepsilon_n \in (|\mathfrak{A}|,|\mathfrak{B}|)$ respectively, replacing the letters A and B by, respectively, Gothic letters $\mathfrak{A}$ and $\mathfrak{B}$.

The first paper concerned with characterizing the summability factors in a sequence was the work of Bosanquet [8], who characterized the summability factors in a sequence of the type (C^α, C^β) when $\alpha, \beta = 0, 1, \cdots$ and $X = Y = \mathbf{C}$. An analogous problem for absolute summability was solved by Tyler [15]. For the matrix of the Euler–Knopp method E_λ, i.e.,

$$a_{nk} = \binom{n}{k} \lambda^k (1-\lambda)^{n-k},$$

the convergence factors in a sequence of the types (E_λ, E) and $(|E_\lambda|,|E|)$ were characterized by Espenberg [10] for $X = Y = \mathbf{C}$, who obtained also in the particular case $\varepsilon_n = x^n$ the summability factors in a sequence of types (E_λ, E_μ) and $(|E_\lambda|,|E_\mu|)$ for $\lambda, \mu \neq 0$.

If $A = P$ is the matrix of the method $P = (R, p_n)$ of the weighted means of Riesz, i.e., if

$$a_{nk} = p_k / P_n, \qquad P_n = p_0 + \cdots + p_n \neq 0,$$

where (p_n) is an arbitrary sequence of complex numbers $p_n \neq 0$, the problems of characterizing summability factors were solved in the paper of Kangro and Tynnov [13]. In their paper the summability factors in a sequence of the types (P, B) and $(|P|,|B|)$ are characterized for arbitrary matrix B and arbitrary Banach spaces X and Y. To generalize the result of Kangro and Tynnov about $\varepsilon_n \in (|\mathfrak{P}|,|\mathfrak{B}|)$ the author [5] characterized the summability factors in a sequence of the type $(|A^\alpha|,|B|)$ and hence of the type $(|C^\alpha|,|B|)$, for $\alpha = 0, 1, \cdots$, when A^α is a normal matrix (i.e., a triangular matrix with $a_{nn} \neq 0$), where its inverse matrix $(\xi_{nk}) = (a_{nk})^{-1}$ has $\alpha + 1$ non-zero diagonals, i.e., $\xi_{nk} = 0$ at $k < n - \alpha$. Convergence factors in a sequence of the types (A^α, E) and $(|A^\alpha|,|E|)$ were characterized by Abel' and Türnpu ([1], theorem 11).

From the applications of the summability factors in a sequence it should be noted that Bosanquet ([6], lemma 6; and [7], lemma 7) and Kangro and Tynnov ([13], §3) characterized summability factors in series. Petersen [14] exploited them for obtaining necessary and sufficient conditions for summability. Kangro

([12], theorem 2) and the author ([4], theorem 27.2) employed summability factors in the problem of weak Tauberian conditions.

In the present paper we characterize summability factors in a sequence of the type $(|C^\alpha|, |B|)$, where α is a non-negative number, under certain conditions on the arbitrary triangular matrix B. In particular, we give a generalization of the result of Tyler for non-negative α and $B = C^\beta$ where β satisfies $-1 < \operatorname{Re} \beta < 2$. Also we can characterize the summability factors in a sequence of the types $(|C^\alpha|, |P|)$ and $(|C^\alpha|, |Q|)$, where $P = (R, p_n)$ and Q is the matrix of the Woronoi–Nörlund method (WN, q_n), and if P and Q satisfy some conditions.

Let B be a normal matrix. For the matrix B we denote

$$b_k = \sum_{n=k}^{\infty} |\bar{b}_{nk}|, \qquad b'_k = \sum_{n=k}^{\infty} |\Delta \bar{b}_{nk}|,$$

where $\Delta \bar{b}_{nk} = \bar{b}_{nk} - \bar{b}_{n,k+1}$.

If we denote by v'_n the B-means in the series to series form of the sequence $(\varepsilon_n U_n)$, that is

$$v'_n = \sum_{v=0}^{n} \bar{b}_{nv} \varepsilon_v U_v$$

and applying the inverse transformation of (2), we obtain†

$$v'_n = \sum_{k=0}^{n} \gamma_{nk} u'_k \tag{3}$$

where

$$\gamma_{nk} = \sum_{\nu=k}^{n} \bar{b}_{n\nu} \bar{\xi}_{\nu k} \varepsilon_\nu$$

and $(\bar{\xi}_{nk}) = (\bar{a}_{nk})^{-1}$. Applying to the series to series transformation (3) the theorem of Kangro (see [11], theorem 4), which is the generalization for Banach spaces of the theorem of Knopp–Lorentz ([4], theorem 4.1), we obtain:

We have $\varepsilon_n \in (|\mathfrak{A}|, |\mathfrak{B}|)$ if and only if for each $k = 0, 1, \cdots$ and each $x \in X$ the following condition holds:

$$\sum_{n=k}^{\infty} \| \gamma_{nk} x \| = O(\| x \|). \tag{4}$$

† It is more precise to designate $\bar{\gamma}_{nk}$ instead of γ_{nk} because (3) is a matrix transformation in the series to series form (cf. [4], p. 9). The same can be said concerning λ_{nk}, π_{nk}, ρ_{nk}, σ_{nk} etc.

Since for the convergence method E we have $\bar{\xi}_{nk} = 1$ for $0 \leqq k \leqq n$, then from (4) we obtain (cf. [4], theorem 26.4) directly

THEOREM 1. *In order that* $\varepsilon_n \in (|\mathfrak{E}|, |\mathfrak{B}|)$ *it is necessary and sufficient that*† *for every* $x \in X$

$$(\varepsilon_n x) \in |B|' \tag{5}$$

and

$$\sum_{n=k}^{\infty} \left\| \sum_{\nu=0}^{k-1} \bar{b}_{n\nu} \varepsilon_\nu x \right\| = O(\|x\|). \tag{6}$$

Since $|C^\alpha| \supset |E|$ at $\alpha > 0$ and $C^0 = E$, then the conditions (5) and (6) are necessary also for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{B}|)$ when $\alpha \geqq 0$.

It is known (see the formula (9.8) in [4]) that

$$\bar{\xi}_{nk} = \sum_{s=k}^{n} \xi_{ns}, \tag{7}$$

where (ξ_{nk}) is the inverse of the matrix of the transformation (1). For the Cesàro mean

$$\xi_{nk} = A_k^\alpha A_{n-k}^{-\alpha-1},$$

and hence from (7), we obtain for $0 \leqq k \leqq n$

$$\bar{\xi}_{nk} = 1 - \sum_{s=0}^{k-1} A_s^\alpha A_{n-s}^{-\alpha-1}. \tag{8}$$

Now it is possible to prove

THEOREM 2. *If* $\alpha \geqq 1$ *and* B *satisfies the condition*

$$b_k = O(b_{kk}) \tag{9}$$

then necessary and sufficient conditions for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{B}|)$ *are* (5), (6) *and*

$$\|\varepsilon_k\| = O(k^{-\alpha}/b_{kk}). \tag{10}$$

PROOF. The necessity of condition (10) follows directly from (4) if we consider the term $n = k$ since

† The condition (5) designates that the sequence $(\varepsilon_n x)$ is absolutely B-summable, that is, taking into account a result of Bosanquet–Kestelman (cf. [9], p. 97), that for each $x \in X$

$$\sum_{n=0}^{\infty} \left\| \sum_{\nu=0}^{n} \bar{b}_{n\nu} \varepsilon_\nu x \right\| = O(\|x\|).$$

$$\gamma_{kk} = \bar{b}_{kk}\bar{\xi}_{kk}\varepsilon_k = (b_{kk}/a_{kk})\varepsilon_k.$$

In order to prove the sufficiency part of the theorem we must prove that condition (4) is satisfied. Taking into account (8), we have

$$\gamma_{nk} = \sum_{\nu=k}^{n} \bar{b}_{n\nu}\varepsilon_\nu - \lambda_{nk}, \tag{11}$$

where

$$\lambda_{nk} = \sum_{\nu=k}^{n} \bar{b}_{n\nu}\left(\sum_{s=0}^{k-1} A_s^{\alpha} A_{\nu-s}^{-\alpha-1}\right)\varepsilon_\nu.$$

Applying consecutively conditions (10) and (9), we discover

$$\sum_{n=k}^{\infty} \|\lambda_{nk}x\| = \sum_{n=k}^{n}\sum_{\nu=k}^{n} |\bar{b}_{n\nu}|\, O(\|x\|\nu^{-\alpha}/b_{\nu\nu}) \sum_{s=0}^{k-1} A_s^{\alpha}|A_{\nu-s}^{-\alpha-1}|$$

$$= \sum_{s=0}^{k-1} A_s^{\alpha} \sum_{\nu=k}^{\infty} |A_{\nu-s}^{-\alpha-1}|\, O(\|x\|\nu^{-\alpha}/b_{\nu\nu})b_\nu$$

$$= O(\|x\|k^{-\alpha}) \sum_{s=0}^{k-1} A_s^{\alpha} \sum_{\nu=k-s}^{\infty} |A_\nu^{-\alpha-1}| = O(\|x\|)$$

because $\alpha \geqq 1$. Now in view of the above and the partition (11), condition (4) follows from (9), (10), (5) and (6) by appealing to the proof of Theorem 1.

Condition (9) of Theorem 2 is satisfied by many methods B, for example, by the methods (R, p_n), that preserve absolute convergence, the method (WN, q_n) with $0 \leqq q_n \downarrow$, the method of Bernstein–Rogonsinki and others (cf. [2], §5).

Applying Theorem 2 to the method $P = (R, p_n)$, we obtain

COROLLARY 1. *Let $\alpha > 1$. If the method P preserves absolute convergence, then necessary and sufficient conditions for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{P}|)$ are $(\varepsilon_n x) \in |P|'$ for every $x \in X$ and*

$$\|\varepsilon_k\| = O(P_k k^{-\alpha}/p_k). \tag{12}$$

Indeed if $B = P$ we have

$$\bar{b}_{n\nu} = -p_\nu p_n/(P_n P_{n-1})$$

(see formulas (17.3) and (8.9) in [4]) and condition (6) of Theorem 2 follows from (12), since

$$\sum_{n=k}^{\infty} \left\| \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{k-1} p_\nu \varepsilon_\nu x \right\| = O(\|x\|) \sum_{n=k}^{\infty} \left| \frac{P_{k-1}p_n}{P_n P_{n-1}} \right| \sum_{\nu=0}^{k-1} (\nu+1)^{-\alpha} \left| \frac{P_\nu}{P_{k-1}} \right|$$

$$= O(\|x\|),$$

because P preserves absolute convergence (see [4], theorem 17.2).

Applying Theorem 2 to the method H of harmonic means $(WN, (n+1)^{-1})$ we obtain

COROLLARY 2. *If $\alpha \geqq 1$, then necessary and sufficient conditions for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{H}|)$ are $(\varepsilon_n x) \in |H|'$ for every $x \in X$ and*

$$\|\varepsilon_k\| = O(k^{-\alpha} \ln k). \tag{13}$$

Indeed if $B = H$ we have $\bar{b}_{n\nu} = \bar{\Delta}(q_{n-\nu}/Q_n)$, where $q_n = (n+1)^{-1}$ and $Q_n = 1 + 1/2 + \cdots + (n+1)^{-1} = \ln(n+2) + O(1)$. Therefore

$$\sum_{\nu=0}^{k-1} \bar{b}_{n\nu}\varepsilon_\nu = \zeta_{nk}^1 + \zeta_{nk}^2,$$

where $\zeta_{nk}^1 = \bar{\Delta}(1/Q_n) \cdot \sum_{\nu=0}^{k-1} q_{n-\nu}\varepsilon_\nu$ and $\zeta_{nk}^2 = (1/Q_{n-1})\sum_{\nu=0}^{k-1} \bar{\Delta} q_{n-\nu}\varepsilon_\nu$.

From condition (13) we obtain $\|\varepsilon_k\| = O(q_k Q_k)$ and hence

$$\sum_{n=k}^{\infty} \|\zeta_{nk}^1\| = O(Q_k^2) \sum_{n=k}^{\infty} |\bar{\Delta}(1/Q_n)| q_{n-k} = O(1) \sum_{n=k}^{\infty} q_n q_{n-k} = O(1)$$

and, denoting $K = [k/2]$, we obtain also

$$\begin{aligned} \sum_{n=k}^{\infty} \|\zeta_{nk}^2\| &= O(1) \sum_{\nu=0}^{k-1} (\nu+1)^{-1} Q_\nu \sum_{n=k}^{\infty} q_{n-\nu} q_{n-1-\nu}/Q_{n-1} \\ &= O(1) \sum_{\nu=0}^{k-1} q_\nu q_{k-\nu} = O(q_k)\left(\sum_{\nu=0}^{K} q_\nu + \sum_{\nu=K}^{k} q_{k-\nu}\right) \\ &= O(q_k Q_k) = O(1). \end{aligned}$$

However the method of Cesàro C^β satisfies condition (9) only when $-1 < \beta \leqq 1$ and for complex β at $-1 < \operatorname{Re}\beta < 1$ (see [4], p. 188). To cover this case we prove another theorem, in which condition (9) is replaced by a weaker condition

$$b'_k = O(b_{kk}). \tag{14}$$

The method (WN, q_n) with $q_n > 0$ satisfies condition (14) if Δq_n is non-decreasing and bounded (see [5], p. 39). Condition (14) is also satisfied by any method GC^1, where G is an arbitrary normal method of summability for which (14) is fulfilled.

THEOREM 3. *Let $\alpha \geqq 2$. If B satisfies condition* (14) *and uniformly for all $k < l$*

$$l^{-\alpha}/b_{ll} = O(k^{-\alpha}/b_{kk}), \tag{15}$$

then for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{B}|)$, *sufficient and (if* $b_k = O(b_{k+1})$ *and*

$$\sum_{\kappa=0}^{k} (\kappa+1)^{-1}/b_\kappa = O(1/b_k) \tag{16}$$

also) necessary conditions are: (5), (6), (10) *and for every* $x \in X$ *also*

$$b_k \left\| \sum_{\kappa=0}^{k-1} A_\kappa^\alpha \Delta^\alpha \varepsilon_\kappa x \right\| = O(\|x\|) \tag{17}$$

and

$$b_k \left\| \sum_{\kappa=0}^{k-1} \varepsilon_\kappa x \right\| = O(\|x\|). \tag{18}$$

PROOF. Applying to the operator γ_{nk} partial summation and equation (8) we get

$$\gamma_{nk} = \sum_{\nu=k}^{n} \Delta\bar{b}_{n\nu} \cdot \sum_{\kappa=k}^{\nu} \bar{\xi}_{\kappa k}\varepsilon_\kappa = \pi_{nk} - \rho_{nk} + \sigma_{nk} + \tau_{nk}, \tag{19}$$

where

$$\pi_{nk} = \sum_{\kappa=k}^{n} \sum_{\nu=\kappa}^{n} \Delta\bar{b}_{n\nu}\varepsilon_\kappa = \sum_{\kappa=k}^{n} \bar{b}_{n\kappa}\varepsilon_\kappa,$$

$$\rho_{nk} = \sum_{\nu=k}^{n} \Delta\bar{b}_{n\nu} \cdot \sum_{s=0}^{k-1} \sum_{\kappa=s}^{\infty} \xi_{\kappa s}\varepsilon_\kappa = \bar{b}_{nk} \sum_{s=0}^{k-1} A_s^\alpha \Delta^\alpha \varepsilon_s,$$

$$\sigma_{nk} = \sum_{\nu=k}^{n} \Delta\bar{b}_{n\nu} \cdot \sum_{s=0}^{k-1} A_s^\alpha \sum_{\kappa=s}^{k-1} A_{\kappa-s}^{-\alpha-1}\varepsilon_\kappa = \bar{b}_{nk} \sum_{\kappa=0}^{k-1} \varepsilon_\kappa,$$

$$\tau_{nk} = \sum_{\nu=k}^{n} \Delta\bar{b}_{n\nu} \cdot \sum_{s=0}^{k-1} A_s^\alpha \sum_{\kappa=\nu+1}^{\infty} A_{\kappa-s}^{-\alpha-1}\varepsilon_\kappa.$$

From conditions (14), (10) and (15) follows

$$\begin{aligned}
\sum_{n=k}^{\infty} \|\tau_{nk}\| &\leqq \sum_{s=0}^{k-1} A_s^\alpha \sum_{\nu=k}^{\infty} b'_\nu \sum_{\kappa=\nu+1}^{\infty} |A_{\kappa-s}^{-\alpha-1}| \, \|\varepsilon_\kappa\| \\
&= \sum_{s=0}^{k-1} A_s^\alpha \sum_{\nu=k}^{\infty} b'_\nu O(\nu^{-\alpha}/b_{\nu\nu}) \sum_{\kappa=\nu-s}^{\infty} |A_\kappa^{-\alpha-1}| \\
&= O(1) \sum_{s=0}^{k-1} (s+1)^\alpha \sum_{\nu=k}^{\infty} \nu^{-\alpha}(\nu-s)^{-\alpha} \\
&= O(k^{-\alpha}) \sum_{s=0}^{k-1} (s+1)^\alpha (k-s)^{1-\alpha} = O(1)
\end{aligned}$$

if $\alpha > 2$. In case that $\alpha = 2$, then $A_\kappa^{-\alpha-1} = 0$ for $\kappa > 2$, and thus

$$\sum_{n=k}^{\infty} \|\tau_{nk}\| \leqq A_{k-1}^{2} b_k' |A_2^{-3}| \|\varepsilon_{k+1}\| = O(1)$$

because from (15) follows the condition $b_{kk} = O(b_{k+1,k+1})$ and therefore from (10) we get $\|\varepsilon_{k+1}\| = O(k^{-\alpha}/b_{kk})$. The evaluation of the addend π_{nk} in the partition (19) we obtain from the necessary conditions (5) and (6). Furthermore it is proved (see [5], p. 41) that for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{B}|)$ under the restriction $b_k = O(b_{k+1})$ it is necessary that $b_k \|\Delta^{\alpha-1}\varepsilon_k\| = O(k^{-\alpha})$ if α is an integer. Using this condition and applying partial summation and (16), we deduce

$$\begin{aligned}\sum_{n=k}^{\infty} \|\rho_{nk}\| &= b_k \left\| \sum_{s=0}^{k} A_s^{\alpha-1}\Delta^{\alpha-1}\varepsilon_s - A_k^{\alpha}\Delta^{\alpha-1}\varepsilon_k \right\| \\ &= O(b_k) \sum_{s=0}^{k} (s+1)^{-1}/b_s + O(1) = O(1).\end{aligned}$$

Comparing the expression σ_{nk} with condition (18), and using the evaluations obtained for the operators π_{nk}, ρ_{nk} and τ_{nk}, we get the necessity of condition (18) for $\varepsilon_n \in (|\mathfrak{C}^{[\alpha]}|, |\mathfrak{B}|)$, and from here also for $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{B}|)$, in view of the inclusion $|C^{[\alpha]}| \subset |C^\alpha|$. Now, in view of the necessity of conditions (5), (6), (10) and (18), from the partition (19) and condition (4), follows the necessity of (17) if we compare this condition with the expression ρ_{nk}. Conversely from conditions (5), (6), (10), (17) and (18) taking into account the restrictions (14) and (15) and the partition (19) follows condition (4). This completes the proof of Theorem 3.

Let us adapt Theorems 1–3 to the case $B = C^\beta$. For C^β, condition (14) of Theorem 3, and hence condition (15) also, holds for all β with $-1 < \beta \leqq 2$ or $-1 < \operatorname{Re}\beta < 2$ (see [5], pp. 41–42). For the method C^β we have $C_1\nu_k \leqq b_k \leqq C_2\nu_k$ (see [3], p. 171), where C_1 and C_2 are positive constants and

$$\nu_k = \begin{cases} (k+1)^{-\operatorname{Re}\beta} & \text{for } -1<\beta\leqq 1 \text{ or } \quad -1<\operatorname{Re}\beta<1, \\ (k+1)^{-1}\ln(k+2) & \text{for } \qquad \beta\neq 1 \text{ with } \qquad \operatorname{Re}\beta = 1, \\ (k+1)^{-1} & \text{for } \qquad \beta \geqq 1 \text{ or } \qquad \operatorname{Re}\beta > 1. \end{cases}$$

From here condition (16) is fulfilled if $\operatorname{Re}\beta > 0$ (see for example [16], p. 192). Furthermore it is proved (see [5], p. 43), that in the case $B = C^\beta$, the condition (6) follows from (18), and (18) is necessary. Consequently from Theorems 1–3 follows

COROLLARY 3. *If $-1 < \beta \leqq 2$ or $-1 < \operatorname{Re}\beta < 2$, then necessary and sufficient conditions for $\varepsilon_n \in (|\mathfrak{C}|, |\mathfrak{C}^\beta|)$ are*

(20) $$(\varepsilon_n x) \in |C^\beta|'$$

and

$$\nu_k \left\| \sum_{\kappa=0}^{k-1} \varepsilon_\kappa x \right\| = O(\|x\|) \tag{21}$$

for every $x \in X$. *If* $\alpha \geqq 1$ *and* $-1 < \beta \leqq 1$ *or* $-1 < \operatorname{Re} \beta < 1$, *then necessary and sufficient conditions for* $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{C}^\beta|)$ *are* (20), (21) *and*

$$\|\varepsilon_k\| = O(k^{\beta-\alpha}). \tag{22}$$

If $\alpha \geqq 2$ *and* $1 < \beta \leqq 2$ *or* $1 \leqq \operatorname{Re} \beta < 2$, *then the necessary and sufficient conditions for* $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{C}^\beta|)$ *are* (22) *and for every* $x \in X$, *also* (20), (21) *and*

$$\nu_k \left\| \sum_{\kappa=0}^{k-1} A_\kappa^\alpha \Delta^\alpha \varepsilon_\kappa x \right\| = O(\|x\|).$$

For the case $0 < \alpha < 1$ we will be considering only absolute convergence factors.

THEOREM 4. *If* $0 < \alpha < 1$ *then necessary and sufficient conditions for* $\varepsilon_n \in (|\mathfrak{C}^\alpha|, |\mathfrak{C}|)$ *are*

$$\|\varepsilon_k\| = O(k^{\alpha}) \tag{23}$$

and for every $x \in X$ *also*

$$\sum_{n=0}^{\infty} \|\bar{\Delta}\varepsilon_n x\| = O(\|x\|) \tag{24}$$

and

$$\sum_{n=k}^{\infty} (n+1-k)^{-\alpha} \|\bar{\Delta}\varepsilon_n x\| = O(k^{-\alpha}\|x\|). \tag{25}$$

PROOF. Since $B = E$ conditions (5) and (10) become (24) and (23) respectively, but (6) reduces to $\|\varepsilon_{k-1}x\| = O(\|x\|)$ and follows from (23). Taking into account the proof of Theorem 2 we must consider the operator

$$\lambda_{nk} = \sum_{s=0}^{k-1} A_s^\alpha A_{n-s}^{-\alpha-2} \varepsilon_{n-1} + \sum_{s=0}^{k-1} A_s^\alpha A_{n-s}^{-\alpha-1} \bar{\Delta}\varepsilon_n.$$

Applying partial summation, we obtain

$$\lambda_{nk} = \varphi'_{nk} - \varphi''_{nk} + \psi'_{nk} - \psi''_{nk},$$

where

$$\varphi'_{nk} = \sum_{\kappa=0}^{k-1} A_\kappa^{\alpha-1} A_{n-\kappa}^{-\alpha-1} \varepsilon_{n-1}, \qquad \varphi''_{nk} = A_{k-1}^{\alpha} A_{n-k}^{-\alpha-1} \varepsilon_{n-1},$$

$$\psi'_{nk} = \sum_{\kappa=0}^{k-1} A_\kappa^{\alpha-1} A_{n-\kappa}^{-\alpha} \bar{\Delta} \varepsilon_n, \qquad \psi''_{nk} = A_{k-1}^{\alpha} A_{n-k}^{-\alpha} \bar{\Delta} \varepsilon_n.$$

Further, for $n > 0$,

$$\varphi'_{nk} = \sum_{\kappa=k}^{n} A_\kappa^{\alpha-1} A_{n-\kappa}^{-\alpha-1} \varepsilon_{n-1} = \sum_{\kappa=0}^{n-k} A_{n-\kappa}^{\alpha-1} A_\kappa^{-\alpha-1} \varepsilon_{n-1}.$$

By the formulas of Bosanquet (see [4], formula (15.20), or [7], pp. 487–488) and Chow (see formula (15.19) in [4]) from condition (23) follows

$$\sum_{n=k}^{\infty} \|\varphi'_{nk}\| = O(1) \sum_{n=k}^{\infty} \|\varepsilon_{n-1}\| A_n^{\alpha-1} A_{n-k}^{-\alpha} = O(1) \sum_{n=k}^{\infty} A_{n-k}^{-\alpha} / A_n^1 = O(k^{-\alpha}) = O(1).$$

Condition (23) implies

$$\sum_{n=k}^{\infty} \|\varphi''_{nk}\| = O(k^\alpha) \sum_{n=k}^{\infty} O(n^{-\alpha}) |A_{n-k}^{\alpha-1}| = O(1) \sum_{n=k}^{\infty} |A_{n-k}^{-\alpha-1}| = O(1).$$

From condition (24) we deduce for every $x \in X$

$$\sum_{n=k}^{\infty} \|\psi'_{nk} x\| \leqq \sum_{n=k}^{\infty} \|\bar{\Delta} \varepsilon_n x\| \sum_{\kappa=0}^{n} A_\kappa^{\alpha-1} A_{n-\kappa}^{-\alpha} = \sum_{n=k}^{\infty} \|\bar{\Delta} \varepsilon_n x\| = O(\|x\|).$$

The evaluation of the operator ψ''_{nk} reduces to condition (25).

REFERENCES

1. M. Abel' and H. Tjurnpu, *On ψ-convergence factors*, Tartu Ülikooli Toimetised **206** (1967), 106–121 (in Russian); MR **41**#7336.

2. S. Baron, *Theorems on summability factors for A^α methods*, Tartu Ülikooli Toimetised **253** (1970), 165–178 (in Russian); MR **44** #680.

3. S. Baron, *The local property of the absolute summability of differentiated Fourier series*, Tartu Ülikooli Toimetised **281** (1971), 157–177 (in Russian); MR **46** #9630.

4. S. Baron, *Introduction to the Theory of Summability of Series* (Second Edition, Corrected and Supplemented), Tallinn, "Valgus", 1977 (in Russian); MR **81j**:40007.

5. S. Baron, *Absolute summability factors in a sequence*, Tartu Ülikooli Toimetized **504** (1981), 35–47 (in Russian).

6. L. S. Bosanquet, *Note on convergence and summability factors, II*, Proc. Lond. Math. Soc. (2) **50** (1949), 295–304.

7. L. S. Bosanquet, *Note on convergence and summability factors, III*, Proc. Lond. Math. Soc. (2) **50** (1949), 482–496.

8. L. S. Bosanquet, *On convergence and summability factors in a sequence*, Mathematika **1** (1954), 24–44.

9. L. S. Bosanquet and H. Kestelman, *The absolute convergence of series of integrals*, Proc. Lond. Math. Soc. (2) **45** (1939), 88–97.

10. H. Espenberg, *Summability factors in a sequence for the Euler–Knopp method*, Sb. Naučn. Trudov Est. Selsk.-Hoz. Akad. **31** (1963), 73–81 (in Russian).

11. G. Kangro, *Über Matrixtransformationen von Folgen in Banachschen Räumen*, Izv. Akad. Nauk Eston. SSR. Ser. Tehn. Fiz-Mat. Nauk **5** (1956), 108–128 (in Russian); MR **20** #4121.

12. G. Kangro, *The weakening of Tauber's conditions*, Eesti NSV Tead. Akad. Toimetised, Füüs.-Mat. **19** (1970), 24–33 (in Russian); MR **43** #769.

13. G. Kangro and M. Tynnov, *Summability factors in sequences for the weighted Riesz means*, Tartu Ülikooli Toimetised **102** (1961), 249–262 (in Russian); MR **26** #1653.

14. G. M. Petersen, *Consistency of summation matrices for unbounded sequences*, Q. J. Math. Oxford Ser. 2, **14** (1963), 161–169; MR **27** #3974.

15. B. Tyler, *Absolute convergence and summability factors in a sequence*, J. Lond. Math. Soc. **33** (1958), 341–351; MR **20** #5378.

16. A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge, 1968.

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